• So far we have discussed equations having a one-dimensional state space i.e. the state is determined by a single number.

• The state is essentially the initial information needed for a dynamical system model to operate and respond unambiguously with an orbit.

• In partial differential equations, the state space may be infinitely dimensional.

• For a vibrating string, modeled by a partial differential equation, the state is the entire real-valued function describing the shape of the string.

• It is very rare to find a dynamical system that has an explicit formula that described the evolution of the state’s future (like the projectile equation)

• Often we have equations describing the next state in terms of the previous and/or current state.

• A formula that described the new state in terms of the previous one is known as a map.

• A differential equation is in fact a formula that describes the instantaneous rate of change of the current state, in terms of the current state.
• A system of two masses interacting through gravitational forces is analytically solvable giving orbital motion on the center of gravity.

• In other words, the motion of the masses can be derived as a function of time.

• Having three masses interacting gravitationally is not solvable. This is known as the 3-body problem.

• This problem has an 18-dimensional state space: the position and velocity of each.

• Nowadays we know there is no exact formula and one uses numerical techniques to approximate.

• Before the non-existence of the formula was known, a competition was held on 1889 to find the solution to this problem: the winner was Henri Poincare.

• He made two assumptions; the third mass was really small and the first two moved in a circle around center of mass.

• Different initial conditions lead to very different trajectories; some lead to the small body escaping, others in a collision path and others in a path that seems unpredictable.

• This is known as a planer restricted three-body problem.
• Poincare discovered that instead of analysing the entire trajectory, one could find most of the important information in the points in which the trajectory passed through a two-dimensional plane.

• The order of the intersection points defines a plane map.

• One can trace a trajectory X whereby the plane S is defined by $x_3=$constant.

• Each time the trajectory C pierces S in a down-ward direction (ex. A and B) we record the point of piercing.

• Let A be the k’th piercing and B the (k+1)’th piercing, then a Poincare map is a two-dimensional map G such that $G(A) = B$.

• This is similar to a time-T map since it is stroboscopic, yet the Poincare map records at variable time intervals.

• While we are studying a plane, other shapes may be considered.

• This surface is known as a surface of section.
• Given A, one can solve the differential equations with A as the initial condition.

• The solution can be iterated until the next downward piercing of S (i.e. at B)

• This ensures that G is well-defined since B is completed defined by A.

• In general, one can reduce a k-dimensional, continuous-time dynamical system to a (k-1) dimensional Poincare map.

• Most of the dynamical behavior of C can be found in G, for example the trajectory C is periodic if G has a periodic orbit.

• The reason that led Poincare to these constructs is beyond this course, yet it is common to examine chaotic systems in this way.

• Henon found that complicated behaviour of differential equations can also be found in the following Henon map

\[ f(x, y) = (a - x^2 + by, x) \]  

(1)

• Using \( a = 1.28 \) and \( b = -0.3 \) we find that some initial conditions converge to a period-two sink or simply diverge to infinity.

• It is common to plot those points that diverge to infinity for a range of values.
• We can say that the white points belong to the basin of period-2 sink and the black point to the basin of infinity.

• Setting \( a = 1.4 \) and \( b = -0.3 \) we get something totally different. Apart from the fact that the period-2 sink has moved a lot, we do not get well defined boundaries.

• This is a fractal structure (See Fig. 2.3b).

• **Exercise:** Find what the period-2 sinks are in the above equations and also plot the fractal structure mentioned above.

• Let’s image a pendulum suspended by a rigid rod under the force of gravity. In addition, assume that it is a friction-less system.

• The differential equation governing this motion is

\[
ml\ddot{\theta} = F = -mg \sin \theta \tag{2}
\]

• Thus this is a two-dimensional state space equation since one needs to know two initial values \( \theta(0) \) and \( \dot{\theta}(0) \) for \( t = 0 \).

• Note that knowing \( \theta(0) = 0 \) alone is not enough to predict the motion of the pendulum and the same if we know \( \dot{\theta} = 0 \) alone.

• To simplify thing we will set \( l = g \) and we will set a damping term \( -c\dot{\theta} \) and a periodic external force \( p \sin t \).
• End result is a forced damped pendulum

\[ \ddot{\theta} = -c\dot{\theta} - \sin \theta + p \sin t \]  

(3)

• Since the force is periodic with period $2\pi$, then if $\theta(t)$ is a solution, then so is $\theta(t + 2\pi N)$ (remember that the pendulum at $\theta + 2\pi$ is at the same position as at $\theta$).

• Define $u(t) = \theta(t + 2\pi)$. Obviously $\dot{u}(t) = \dot{\theta}(t)$ and $\ddot{u}(t) = \ddot{\theta}(t)$.

• Since $\theta(t)$ is a solution, then

\[ \ddot{\theta}(t+2\pi) = -c\dot{\theta}(t+2\pi) - \sin \theta(t+2\pi) + p \sin(t+2\pi) \]  

(4)

• But $\sin t = \sin(t + 2\pi)$ thus

\[ \ddot{u}(t) = -c\dot{u}(t) - \sin u(t) + p \sin t \]  

(5)

Thus $u(t)$ is also a solution to (3)

• We can conclude that the time-$2\pi$ map of the forced damped pendulum is well defined.

• In other words, if $(\theta_1, \dot{\theta}_1)$ is the result of starting with initial conditions $(\theta_0, \dot{\theta}_0)$ at $t = 0$ then we would get the same results if we start with $(\theta_0, \dot{\theta}_0)$ at $t = 2N\pi$.

• Even though the equation is differential, we can look at the equation at $2\pi$ time units.
• Thus we can define the following

\[ F(\theta_0, \dot{\theta}_0) = (\theta_1, \dot{\theta}_1) \]  

(6)

to be a time-2\pi map F.

• Note that F is not easily solvable and one must iterate through the equation to derive the values from 0 to 2\pi.

• By using \( p = 1.66 \) and \( c = 0.2 \) we get three attractors: one fixed point and two period-two orbits. By drawing different shades for each of the initial conditions, we get a fractal behaviour (Fig 2.5).

• Note that the system has simple low period orbits, yet the boundaries between the three basins is infinitely-layered, or fractal.

• By zooming on particular sections, one finds that the complexity does not decrease, a characteristic of fractal behaviour.

• Plotting using \( c = 0.05 \) and \( p = 2.5 \) we get no sinks (fixed or periodic). Plotting the orbit of the trajectory of the pendulum for a large number of points and then plotting for the next large number of points gives the same pattern.

• Thus there are many fixed points and periodic orbits, yet the behavior is not a simple basin.
• **Exercise to collect:** Plot the basins of the pendulum for \( p = 1.66 \) and \( c = 0.2 \). Show different magnification for each. Plot the orbit for \( c = 0.05 \) and \( p = 2.5 \) for 1 million points. You need to make certain basic assumptions.

• **Def:** The *Euclidean length* of a vector \( v = (x_1, \ldots, x_m) \) in \( \mathbb{R}^m \) is
\[
|v| = \sqrt{x_1^2 + \ldots + x_m^2}.
\]
Let \( p = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m \), and let \( \epsilon \) be a positive number. The \( \epsilon \)-neighborhood \( N_\epsilon(p) \) is the set \( \{v \in \mathbb{R}^m : |v - p| < \epsilon\} \), the set of points within the Euclidean distance \( \epsilon \) of \( p \). We can call \( N_\epsilon(p) \) an \( \epsilon \)-disk centered at \( p \).

• **Def:** Let \( f \) be a map on \( \mathbb{R}^m \) and let \( p \) in \( \mathbb{R}^m \) be a fixed point, that is, \( f(p) = p \). If there is an \( \epsilon > 0 \) such that for all \( v \) in the \( \epsilon \)-neighborhood \( N_\epsilon(p) \), \( \lim_{k \to \infty} f^k(v) = p \), then \( p \) is a sink or attractor. if there is an \( \epsilon \)-neighborhood \( N_\epsilon(p) \) such that each \( v \) in \( N_\epsilon(p) \) except for \( p \) itself eventually maps outside of \( N_\epsilon(p) \), then \( p \) is a source or repeller.

• Fig 2.8 shows typical examples of sinks and sources.

• A new type of fixed point exists known as a *saddle* that cannot exist in one-dimensional maps. Saddles can repel in some directions and attract in others.
• Consider the Henon map

\[ f(x, y) = (-x^2 + 0.4y, x) \]  \hspace{1cm} (7)

• This has two fixed points (0,0) and (-0.6, -0.6). Prove it! The first point is a sink while the second is a saddle.

• To spot saddles, one should draw a disk N and then plot the points of the next iterate for all values in N.

• Saddles like sources are unstable fixed points and thus are sensitive to initial conditions, yet they are very important in describing the dynamics of behaviour.

• Plotting the basin of this map, one finds many points being the basin of the point (0,0) and many more being the basin of infinity. Yet what happens at the border?

• Points at the border converge to the saddle point!! This indicates that saddles are important in deciding the border line of the two basins.

• Considering the Henon map with \( a = 2 \) and \( b = -0.3 \) we find a large region that is the basin of two curves. Then each curve maps to the other on each iteration.

• **Exercise to collect:** Draw the basin shown in Figure 2.11

• See Addendum
What happens to a ball placed on top of a mountain since both the valley and the peak are steady states?

Commonly any point near an unstable steady state will move away and be attracted to a stable steady state or stable periodic steady state.

Consider an initial condition near a source $p$ of map $f$. At the beginning of the orbit, unstable behaviour is displayed.

Exponential separation means that the distance between the orbit point and the source increases at an exponential rate. Each iteration multiplies the separation by $|f'(p)| > 1$. Thus the exponential rate is $|f'(p)|$ per iterate.

The orbit may be attracted to a sink $q$ and the distance between the orbit point and the sink will change by a factor of $|f'(q)| < 1$.

Thus unstable behaviour is transient giving way to a stable behaviour. Yet there are cases where no stable states exist ex $G(x) = 4x(1 - x)$.

Def: A chaotic orbit is one that forever continues to experience the unstable behaviour that an orbit exhibits near the source, but that is not itself fixed or periodic. There is no sink to be attracted to.
• We will use *Lyapunov numbers* to be the average per-step divergence rate of nearby points along the orbit.

• The *Lyapunov exponent* is the natural logarithm of Lyapunov numbers.

• When we have the Lyapunov exponent $> 0$ then we have chaos!

• We saw that if $x_1$ is a fixed point of $f$ (a 1-D map) and $f'(x_1) = a > 1$, then the orbit of $x$ near $x_1$ will separate from $x_1$ at a rate of approximately $a$ per iteration, until the orbit is significantly far from $x_1$.

• For a period point of period $k$, we look at the product of the derivatives of the $k$ points of the orbit. If this product is $A > 1$, then the orbit of each neighbor $x$ of the periodic point $x_1$ separates from $x_1$ at a rate of approximately $A$ after each $k$ iterates.

• This is a cumulative amount of separation since it takes $k$ iterates of the map to separate by a distance $A$. Thus we can describe the average multiplicative rate of separation as $A^{1/k}$ per iterate.

• The Lyapunov number describes this average multiplicative rate of separation of points very close to the fixed point.
Thus a Lyapunov number of 2 for the orbit $x_1$ will mean that the distance between the orbit of $x_1$ and the orbit of a nearby point $x$ doubles each iteration, **on average**.

If $x_1$ was a periodic point of period $k$ this is the same as saying

$$|(f^k)'(x_1)| = |f'(x_1)| \ldots |f'(x_k)| = 2^k$$

Yet we will use this concept for points that are not fixed points or periodic.

A Lyapunov number of $\frac{1}{2}$ would mean that the orbits of $x$ and $x_1$ approach each other rapidly.

**Def:** Let $f$ be a smooth map on the real line $\mathbb{R}$. The **Lyapunov number** $L(x)$ of the orbit \{x_1, x_2, x_3, \ldots\} is defined as

$$L(x_1) = \lim_{n \to \infty} \left( |f'(x_1)| \ldots |f'(x_n)| \right)^{1/n}$$

if this limit exists. The **Lyapunov exponent** $h(x_1)$ is defined as

$$h(x_1) = \lim_{n \to \infty} \left( \frac{1}{n} \left( \ln |f'(x_1)| + \ldots + \ln |f'(x_n)| \right) \right)$$

if this limit exists.

Note that $h$ exists only if $L$ exists and $\ln L = h$.

The above definition applies only for 1-dimensional maps.

Note that Lyapunov numbers are undefined for some orbits. In particular an orbit containing a point $x_i$ with $f'(x_i) = 0$ causes the Lyapunov exponent to be undefined.
• From the definition, the Lyapunov number of a fixed point \( x_1 \) is \( |f'(x_1)| \) and the Lyapunov exponent of the orbit is \( \ln |f'(x_1)| \).

• If \( x_1 \) is a periodic point of period \( k \), then the Lyapunov exponent is
  \[
  h(x_1) = \frac{\ln |f'(x_1)| + \ldots + \ln |f'(x_k)|}{k}
  \]

• Thus for a periodic orbit, the Lyapunov number \( e^{h(x_1)} \) describes the average local stretching on a per-iterate basis, near a point on the orbit.

• **Def:** Let \( f \) be a smooth map. An orbit \( \{x_1, x_2, \ldots, x_n, \ldots\} \) is called asymptotically periodic if it converges to a periodic orbit as \( n \to \infty \). This means that there exists a periodic orbit \( \{y_1, y_2, \ldots, y_k, y_1, y_2, \ldots\} \) such that
  \[
  \lim_{n \to \infty} |x_n - y_n| = 0
  \]

• Thus any orbit that is attracted to a sink is also asymptotically periodic. The extreme case when an orbit falls precisely on a periodic orbit is called **eventually periodic**.

• **Theorem:** Let \( f \) be a map on the real line \( \mathbb{R} \). If the orbit \( \{x_1, x_2, \ldots\} \) of \( f \) satisfies \( f'(x_i) \neq 0 \) for all \( i \) and is asymptotically periodic to the periodic orbit \( \{y_1, y_2, \ldots\} \), then the two orbits have identical Lyapunov exponents, assuming both exist.