• Nature shows behaviour that is not linear
• Equations (differential) that had a solution describe two types of behaviour
  – Steady State
  – Periodic or Quasi-Periodic
• In 1975, scientists realised of another type of motion: *Chaos*
  – Erratic (not quasi-periodic with long periods)
  – Not due to large number of particles
  – Possible in very simple systems
• 2 particles in a box have neither motion A nor B and in the long term unpredictable
• Very small changes in initial motion of particles result in large changes in trajectories
• Starting with random position, ALL directions of travel are likely
  – Dynamical Systems Theory
• In Chaotic systems one cannot determine
  – how long the complex behaviour will continue
  – Predict future behaviour
• Assume \( f(x) \) to be a function representing a population
• Let \( f(x) = 2x \) then we have

\[
x_n = f(x_{n-1}) = 2x_{n-1}
\]  
(1)

where \( n \) represents time and \( x_n \) the population at time \( n \).

• Our system is non-random or non-stochastic
• Defn: Two types of dynamical systems
  – Discrete-Time
  – Continuous-Time: where \(((n-1)-n) \rightarrow 0\)
• Our aim is to find out $f^k(x)$ where $f^k(x) = f(f^{k-1})(x)$

• Our model clearly grows exponentially for ever

• A better model is $g(x) = 2x(1 - x)$ for $0 < x < 1$ where now we have

![Graph 1](image1.png)

• These graphs are not adequate to show evolution with time

• Starting with $x_i = 0.01$ and $x_i = 0.9$ we have

![Graph 2](image2.png)

• This shows that $g(x)$ tends to move always towards $0.5$
• **Def**: A function whose domain space and range space are the same is called a *map*. The *orbit* of $x$ under $f$ is the set of points $\{x, f(x), f^2(x), \ldots\}$. The starting point $x$ for the orbit is the *initial value* of the orbit. A point $p$ is a *fixed point* if $f(p) = p$.

• Thus to find fixed points, one solves the equation $f(x) = x$.

• To plot an orbit one uses a *cobweb plot*. For $g(x)$ where $x_i = 0.1$ and $x_i = 0.7$ we have

One can see that the orbit converges to $g(x) = 0.5$

• **Exercise**: Find the fixed points of $f(x) = 2x$

• **Exercise**: Find expressions for $f^r(x)$ and $g^r(x)$ for $r = 2, 3, 4, 5, 6$

• **Exercise**: From the above derive a general expression in terms of $r$
• **Exercise**: Plot the function \( f(x) = \frac{(3x-x^3)}{2} \)

• **Exercise**: Find the fixed points for this function

• **Exercise**: Draw up the cob-webplot for this function

• **Exercise**: Examine the behaviour of \( f^r(x) \) for \( x_i \) close to 1, 0 and -1.
• **Def:** The *epsilon neighborhood* $N_\epsilon(p)$ is the interval of numbers $\{x \in \mathbb{R} : |x - p| < \epsilon\}. \epsilon$ is usually a small, positive number.

• **Def:** Let $f$ be a map on $\mathbb{R}$ and $p$ be a fixed point of $f$. If there is an $\epsilon > 0$ such that for all $x$ in the epsilon neighborhood $N_\epsilon(p)$, $\lim_{k \to \infty} f^k(x) = p$, then $p$ is a *sink* or an attracting fixed point.

• **Def:** Likewise, if there is an $\epsilon > 0$ such that for all $x$ in the epsilon neighborhood $N_\epsilon(p)$, and $\lim_{k \to \infty} f^k(x)$ maps outside $N_\epsilon(p)$, then $p$ is a *source* or a repelling fixed point.

• **Theorem 1:** Let $f$ be a smooth map on $\mathbb{R}$, and assume that $p$ is a fixed point of $f$
  
  - if $|f'(p)| < 1$, then $p$ is a sink,
  
  - if $|f'(p)| > 1$, then $p$ is a source,

• Prove the above.

• The above merely states that points close to the fixed point are attracted or repelled to $p$. But $\epsilon$ might be extremely small. In fact, sinks usually attract a large number of points.

• **Def:** The set of initial conditions whose orbits converge to the sink is known as the *basin* of the sink.

• **Exercise:** Verify the above theorem for $g(x)$. 

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• What is the basin for fixed point 0.5 in \( g(x) \)?

One can see that the basin is the interval \((0, 1)\). Obviously \( x_i = 0 \) and \( x_i = 1 \) are not basins of this sink (verify!).

• Another method which is more formal is to compare algebraically \( |g(x) - \frac{1}{2}| \) to \( |x - \frac{1}{2}| \).

• Try the above to see that any point \( x \in (0, 1) \) will be decreased towards the fixed point when applied to \( g \).

• **Exercise** Apply the above techniques to \( f(x) = \frac{(3x-x^3)}{2} \).

• Note that we cannot determine the stability of a fixed point where \( f'(p) = 0 \).
• What happens in the equation
\[ h(x) = 3.3x(1 - x) \]  \hspace{1cm} (2)
• The fixed points are \( x = 0 \) and \( x = 0.69 \). Are they attractors or repellers?
• Since they are repellers, what does the orbit do?

\begin{center}
\includegraphics[width=0.5\textwidth]{chaos_diagram.png}
\end{center}

• Looking at the above and calculating the points, one finds the orbit settles to a pattern of alternating between \( p_1 = 0.4794 \) and \( p_2 = 0.8236 \).
• This is typical of an orbit that converges to a period-2 sink \( \{p_1, p_2\} \).
• Two main points are:
  \begin{itemize}
    \item \( h(p_1) = p_2 \) and \( h(p_2) = p_1 \). Thus \( h^2(p_1) = p_1 \). Thus \( p_1 \) is a fixed point of \( h^2 \).
    \item This periodic oscillation is stable and attracts orbits. The pair \( \{p_1, p_2\} \) is a periodic orbit.
  \end{itemize}
• **Def:** We call $p$ a *periodic point of period* $k$ if $f^k(p) = p$ and if $k$ is the smallest such positive integer. The orbit with initial point $p$ is called a *periodic orbit of period* $k$. This is also known as a *period-k orbit*.

• The map $f(x) = -x$ is an interesting example. It has one fixed point and any other point is a period-two point.

• To find period $k$ orbits, one obviously would solve $f^k(x) = x$ for $x$.

• **Exercise:** Find the period 1 and period 2 fixed points of $f(x) = 2x^2 - 5x$.

• **Def:** Let $f$ be a map and assume that $p$ is a period-$k$ point. The period-$k$ orbit of $p$ is a *periodic sink* if $p$ is a sink for the map $f^k$. Likewise the orbit of $p$ is a *periodic source* if $p$ is a source for the map $f^k$.

• The chain rule states

\[(f \circ g)'(x) = f'(g(x))g'(x)\]  
(3)

for $f = g$

\[(f^2)'(x) = f'(f(x))f'(x)\]  
(4)

• Thus the derivative of $f^2$ for a period-2 orbit is simple the product of the derivates of $f$ at the two points in the orbit!
• Using the chain rule one can speak of the stability of period-2 orbits.

• Thus from Theorem 1, if \((f^2)'(p_1) < 1\) then a period-two orbit will be a sink.

• For \(h(x)\), we had a periodic orbit of .4984 and .8236.
  \[
  (h^2)'(p_1) = h'(p_1)h'(p_2) = (h^2)'(p_2)
  \]
  \[
  h'(0.4794)h'(8236) < 1
  \]
  Thus the period-2 orbit is a sink.

• Exercise: Examine \(g(x) = ax(1 - x)\) for \(a = 3.5\). Produce the cobweb plots and verify what period orbit this function represent. Verify if the ensuing orbit is a sink or a source.

• Theorem 2 The periodic orbit \(\{p_1, ..., p_k\}\) is a sink if
  \[
  |f'(p_k)...f'(p_1)| < 1
  \]
  and a source if
  \[
  |f'(p_k)...f'(p_1)| > 1
  \]

• Exercise: Prove this using the chain rule for \((f^k)'(p_1)\).