1. Consider the grammar $G$ with the following production rules (where $S$, $A$ and $B$ are non-terminal symbols, $a$ and $b$ are terminal symbols and $S$ is the start symbol):

\[
S \rightarrow AS \mid X \\
AX \rightarrow aXb \\
Aa \rightarrow aA \\
X \rightarrow \varepsilon
\]

(a) Prove that $\{A^nS \mid n \in \mathbb{N}\} \subseteq S(G)$.

(b) Hence or otherwise prove that $\{A^nX \mid n \in \mathbb{N}\} \subseteq S(G)$.

(c) Using induction on $n$, prove that for any natural numbers $n$ and $m$, $A^n a X b^m \Rightarrow^* a A^n X b^m$. 


2. (a) Using standard constructions, and showing intermediate constructions, minimise the number of states in the following automaton:

(b) Call the language accepted by the above automaton $L$. Using standard constructions, and showing intermediate constructions, give an automaton which accepts the language $L \cup LL$.

(c) A right-recursive regular grammar is similar to a regular grammar. The production rules of a right-recursive regular grammar must be in the following forms: $A \rightarrow \varepsilon$, $A \rightarrow a$ or $A \rightarrow aB$ (where $A$ and $B$ are any non-terminal symbols and $a$ is a terminal symbol). Formally define a right-recursive regular grammar.

A right-recursive regular language is a language accepted by a right-recursive regular grammar. Give a construction which shows right-recursive regular languages are closed under set union.

3. (a) Using standard constructions, give a regular expression equivalent to the following finite-state automaton:

(b) Given a finite state automaton $M = \langle Q, \Sigma, q_0, t, F \rangle$, we can construct automaton $M' = \langle Q', \Sigma, q_0, t', F \rangle$, where $Q'$ and $t'$ are defined as follows:

\[
Q' = Q \cup \{(q, a) \mid q \in Q, a \in \Sigma\}
\]
\[
t' = t \cup \{(q, a, (q', a)) \mid (q, a, q') \in t\} \cup \{(q, a, q) \mid q \in Q, a \in \Sigma\}
\]

Show your understanding of the above construction by applying it to the automaton constructed in part (a) of this question.