Answer any two questions. Each question carries 50 marks. Students are allowed to use course notes, books and calculators.

1. Consider the grammar $G$ with the following production rules (where $S$ and $A$ are non-terminal symbols, $a$ and $\geq$ are terminal symbols and $S$ is the start symbol):

   $S \rightarrow aSa \mid A \geq$
   $A \rightarrow aA \mid \varepsilon$

   (a) Prove that $\{a^n \mid n \in \mathbb{N}\} \subseteq \{w \mid A \Rightarrow^* w\}$.
   (b) Prove that $\{a^nS^na^n \mid n \in \mathbb{N}\} \subseteq \{w \mid S \Rightarrow^* w\}$.
   (c) Hence or otherwise prove that $\{a^n \geq a^m \mid n \geq m\} \subseteq \mathcal{L}(G)$.

2. (a) Using induction on $n$, prove that $(a^n)^R = a^n$.
   (b) Hence prove that $(a^nba^n)^R = a^nba^n$.
   (c) Using standard constructions, give a pushdown automaton equivalent to the context-free grammar given in question 1.
   (d) Using standard constructions, give a regular grammar equivalent to the following regular expression $(a + ab)^+$. 

---

PTO
3. (a) Consider an automaton $M$ with two states $q_0$ and $q_1$ ($q_0$ is the initial state, and $q_0$ is also the only final state), and the following transition relation:

$\{(q_0, a, q_0), (q_1, a, q_0), (q_0, a, q_1), (q_1, b, q_1)\}$

Using standard constructions, give a regular expression equivalent to $M$.

(b) Given a finite state automaton $M = \langle Q, \Sigma, q_0, t, F \rangle$, we can construct automaton $M' = \langle Q \cup \{\delta\}, \Sigma, \delta, t', \{q_0\}\rangle$, where $t'$ is defined as follows:

$\{(q', a, q) \mid (q, a, q') \in t\} \cup \{(\delta, a, q) \mid \exists q' : Q \cdot q' \in F \land (q', a, q) \in t\}$

It can be proved that $M'$ accepts the reverse of the language accepted by $M$. Show your understanding of the above construction, by applying it to the automaton constructed in part (a) of this question.