1. (a) Formally prove that \( \neg(P \lor Q) \vdash P \iff Q \).

(b) Formally prove that \( (\forall x : X \cdot P) \Rightarrow (\forall x : X \cdot Q) \vdash \exists x : X \cdot P \Rightarrow Q \).

(c) Prove that \( B \subseteq (A \setminus B)^c \).

(d) Prove that, if \( r \in X \leftrightarrow Y \), \( s \in Y \leftrightarrow Z \) are total injective relations, then so is \( r; s \).

(e) We say that a set \( S \) is smaller than a set \( T \) (written as \( S \ll T \)), if there exists a total injective relation from \( S \) to \( T \). Prove that, if \( S \ll T \) and \( T \ll U \), then \( S \ll U \).

(f) Consider the following function defined on natural numbers:

\[
\begin{align*}
\text{half}(0) & \overset{\text{def}}{=} 0 \\
\text{half}(\text{succ}(0)) & \overset{\text{def}}{=} 0 \\
\text{half}(\text{succ}(\text{succ}(n))) & \overset{\text{def}}{=} \text{succ}(\text{half}(n))
\end{align*}
\]

Using the standard definition of addition, inductively prove that for any natural number \( n \), \( \text{half}(n + n) = n \).

2. (a) Formally prove that \( \neg(P \leftrightarrow Q) \vdash P \lor Q \).
(b) Formally prove that $\forall x : X \cdot P \Rightarrow Q \vdash (\exists x : X \cdot P) \Rightarrow (\exists x : X \cdot Q)$.

(c) Prove that if $A \subseteq A \setminus B$, then $A \cap B = \emptyset$.

(d) A black-and-white set of objects of type $T$ is very similar to a set with objects of type $T$, except that every object has exactly one tag: either white or black. Black-and-white sets ignore repetition and order. Formalise this concept, by defining $BW_T$. Define operators: (i) $elem$ (which given an object $x$ of type $T$ and a black-and-white set $S$, returns whether $x$ is in $S$, regardless of colour); (ii) $black$ (which given a black-and-white set $S$, returns all black elements of $S$); (iii) $flip$ (which given a black-and-white set $S$, returns a new set with all colours flipped from white to black, and black to white).

(e) Defining length and $++$ in the standard recursive manner, and reverse in the following manner:

\[
\begin{align*}
\text{reverse}(\text{Nil}) & \overset{\text{def}}{=} \text{Nil} \\
\text{reverse}(\text{Cons} \ (x, \ xs)) & \overset{\text{def}}{=} \text{reverse}(xs) ++ \text{Cons}(x, \text{Nil})
\end{align*}
\]

Inductively prove that for any list $xs$:

$\text{length}(\text{reverse}(xs)) = \text{length}(xs)$