CSA1060: Mathematics of Discrete Structures 8th June 2004

13:00–15:30

Answer any two questions. Each question carries 50 marks. Students are allowed to use course notes, books, calculators and lawn mowers. Theorems and lemmata proved in the lecture notes may be quoted. The mark will contribute towards 70% of the final mark of CSA1060.

1. (a) Prove that \((A \setminus B) \cap (B \setminus A) = \emptyset\).

(b) Formally prove that \(\exists x : X \cdot P(x) \vdash \forall x : X \cdot \exists y : X \cdot P(x) \Rightarrow P(y)\).
   Recall that for any \(x\) and \(y\), \(P(x)[y/x] = P(y)\).

(c) Prove by induction on \(n\), that if relation \(r\) is reflexive, \(r^n\), is also reflexive.

2. (a) Formally prove that \(P \Rightarrow (Q \Rightarrow R) \iff R \lor \neg(P \land Q)\).

(b) Prove that if \(r\) is an reflexive relation, and \(s\) is a symmetric relation, then \(r \circ s\) is symmetric. Give a counter-example to show that \(r \circ s\) need not be reflexive.

(c) Infinite sequences over a set \(X\) can be formalised using total functions from the natural numbers to \(X\).
   
   i. Formalise the set of all infinite sequences over \(X\) (referred to as \(iseq_X\)).

   ii. Define the function \(\text{skip}\), which given an infinite sequence returns the infinite sequence consisting of the first, third, fifth, etc elements of the original sequence.
iii. Define functions which extract the head (first element) and tail (all the list, other than the first element) of an infinite list. Are they total functions?

3. (a) Prove that if $f \in X \leftrightarrow X$ is reflexive and functional, then $f$ is the identity function on $X$ (which we usually refer to as $id_X$).

(b) Prove that $A \cup (A \setminus B) = A$.

(c) Consider the following rules of inference for the Boolean operator $\geq$:

**Rules of introduction:**

\[
\begin{array}{ccc}
 & P & \Rightarrow P \geq Q \\
\hline & \neg Q & \Rightarrow P \geq Q
\end{array}
\]

**Rules of elimination:**

\[
\begin{array}{ccc}
 & P \geq Q, Q & \Rightarrow P \\
\hline & P \geq Q, \neg P & \Rightarrow \neg Q
\end{array}
\]

Formally prove that $P \geq Q \vdash Q \Rightarrow P$ and that $(P \geq Q) \land (Q \geq R) \vdash (P \geq R)$. 