UNIVERSITY OF MALTA
BOARD OF STUDIES FOR INFORMATION TECHNOLOGY/
FACULTY OF SCIENCE
Department of Computer Science & AI
B.Sc. I.T.(Hons.) / B.Sc.(Hons.) – I Year
September 2004 Resit Session

CSA1060: Mathematics of Discrete Structures 9th Sep 2004

09:00–11:30

Answer question 1 and one of questions 2 and 3. Question 1 carries 75 marks, while questions 2 and 3 carry 25 marks each. Students are allowed to use course notes, books, calculators and baseball bats. Theorems and lemmata proved in the lecture notes may be quoted.

1. (a) Prove that \(( P \lor R ) \Rightarrow Q \vdash \neg Q \Rightarrow \neg P.\)

(b) Prove that \(\forall x : X : (P \Rightarrow Q) \vdash (\exists x : X \cdot P) \Rightarrow (\exists x : X \cdot Q).\)

(c) Rigourously prove that if \(P = P \cap Q,\) then \(P \subseteq Q.\)

(d) Rigourously prove that, given a relation \(r\), \(r^{-1} \circ r^{-1} = (r \circ r)^{-1}.\)

[75 marks]

2. A number-labelled graph is a graph where all edges have a number associated with them. Such a graph can be formalised as a pair \(\langle V, E \rangle,\) where \(V\) is the set of vertices of the graph, and \(E \subseteq V \times \mathbb{Z} \times V\) is the set of edges.

(a) The addition of two such graphs is defined as follows:
\[
\langle V_1, E_1 \rangle \oplus \langle V_2, E_2 \rangle \overset{def}{=} \langle V_1 \cup V_2, E \rangle
\]
where \(E\) is defined to be:
\[
E \overset{def}{=} E_1 \cup E_2 \cup \left\{(x, n_1 + n_2, y) \mid (x, n_1, y) \in E_1 \land (x, n_2, y) \in E_2\right\}
\]
Show your understanding of this statement by showing (drawing or writing out the mathematical objects) the graph \(G_1 \oplus G_2,\) where:
\[
G_1 = \langle\{a, b, c\}, \{(a, 1, a), (a, -3, b), (b, 17, a), (c, 3, a)\}\rangle
\]
\[
G_2 = \langle\{a, c, d\}, \{(a, -5, a), (a, 3, c), (c, 7, a)\}\rangle
\]

[75 marks]
(b) The inverse of a number-labelled graph has the same vertices, but has the edges reversed and numbers negated (an edge from $x$ to $y$ with label $n$, becomes an edge from $y$ to $x$ with label $-n$). Define such an operator, in a manner similar to the definition given in part (a) of the question.

(c) A number-labelled graph is said to be transitive if for every pair of edges, one from $x$ to $y$ (with label $n$) and another from $y$ to $z$ (with label $m$), there must exist an edge from $x$ to $z$ with label $n + m$. Give a predicate which is satisfied if and only if $\langle V, E \rangle$ is transitive.

[25 marks]

3. An array of integers of length $n$ can be described mathematically as a function from the number \{1, 2, \ldots n\} to integers:
$$\text{array}_n = 1..n \rightarrow \mathbb{Z}$$

Note that $1..n$ is defined as \{m \mid 1 \leq m \leq n\}.

(a) Explain, in plain English, what it means for an array $a$ (of length $n$) to satisfy the following property:
$$\forall i, j : \mathbb{N} \cdot (i \in 1..n \land j \in 1..n \land i \leq j) \Rightarrow a(i) \leq a(j)$$

(b) Give a predicate which is satisfied by an array $a$ of length $n$ if and only if the array is constant (all the entries have the same value).

(c) Define a function $\text{scale}(a, f)$, which given an array $a$ (of length $n$), and an integer $f$, it returns an array with all entries multiplied by factor $f$.

[25 marks]