Answer one question. Each question carries 100 marks. Students are allowed to use course notes, books and calculators.

1. (a) Consider the language \( L_\Sigma = \{ w\psi(w) \mid w \in \Sigma^* \} \), where \( \psi(w) \) is defined as follows:

\[
\begin{align*}
\psi(\epsilon) & \overset{df}{=} \epsilon \\
\psi(as) & \overset{df}{=} 1a\psi(s)
\end{align*}
\]

Prove that \( L_{\{0\}} \) is not a regular language. Show that it is a context-free language. Is \( L_{\{1\}} \) a context-free language? Is it a regular language?

(b) An \( \alpha \)-stuttering automaton (where \( \alpha \) is an element of the automaton’s alphabet) is a normal non-deterministic finite-state automaton, except that it can always accept any number of repetitions of \( \alpha \) without changing its state. For example, an \( x \)-stuttering automaton \( M \) works just like the normal automaton \( M \), except that it may accept any number of \( x \)s whenever it wants.

As in the case of normal automata, a string is accepted if one can follow transitions from the initial state, leading to a final state.

i. Formalise the notion of an \( \alpha \)-stuttering automaton.

ii. Formalise the configuration of these automata.
iii. Define $C_0$, the initial configuration, and the set $C_F$, the set of final configurations.

iv. Define $C \xrightarrow{a} C'$, meaning that configuration $C'$ can be reached from configuration $C$ reading symbol (not string) $a$.

v. Define $C \xrightarrow{s} C'$, meaning that configuration $C'$ can be reached from configuration $C$ with string $s$.

vi. Hence or otherwise define the language accepted by an $\alpha$-stuttering automaton.

vii. Give a construction to show that every language accepted by an $\alpha$-stuttering automaton is regular.

2. (a) Given two finite state automata $M_1$ and $M_2$ (where $M_i$ is $\langle \Sigma_i, Q_i, I_i, t_i, F_i \rangle$), we can define their asynchronous composition $M_1 \otimes M_2$ to be $\langle \Sigma, Q, I, t, F \rangle$, where:

\[
\begin{align*}
\Sigma & \overset{df}{=} \Sigma_1 \cup \Sigma_2 \\
Q & \overset{df}{=} Q_1 \times Q_2 \\
I & \overset{df}{=} (I_1, I_2) \\
t & \overset{df}{=} \{(q_1, q_2), a, (q_1', q_2') \mid (q_1, a, q_1') \in t_1, \ q_2 \in Q_2\} \\
& \quad \cup \{(q_1, q_2), a, (q_1, q_2') \mid (q_2, a, q_2') \in t_2, \ q_1 \in Q_1\} \\
& \quad \cup \{(q_1, q_2), a, (q_1', q_2') \mid (q_1, a, q_1') \in t_1, \ (q_2, a, q_2') \in t_2\} \\
F & \overset{df}{=} F_1 \times F_2
\end{align*}
\]

Show your understanding of this construct by drawing the asynchronous composition of $M_1$ and $M_2$ defined on the following page:
(b) Consider the following decision problem:

**Car/Bike Travelling Saleswoman (CBTS)**

**Given:** Given a set of $n$ cities $C$, two partial cost functions of travelling from one city to another by car ($cost_c$) or by bike ($cost_b$) (such that $cost_c(c_1, c_2)$ is defined if there is a direct means of travelling from city $c_1$ to city $c_2$, and if defined is the cost of the travelling by car — similarly for $cost_b$), a set of cities from which the saleswoman may switch to a car $C_c$, a set of cities from which the saleswoman may switch to a bike $C_b$ and an amount of money $m$.

**Question:** Starting with a car, is there a way of travelling through all the cities exactly once, returning back to the initial city, and in the process not spending more than the amount of money $m$ (possibly changing means of transport in the process)?

Show that $CBTS \in NP$-complete.

**Hint:** From the input, create an alternative graph, where the nodes encode not only the city in which the saleswoman is but also her means of transport.