1. The traditional way of defining the class of context-sensitive languages is as follows:

A language \( L \) is said to be context-sensitive if there is a grammar \( G \) which generates \( L \) such that every rule in the grammar is of the form:

\[
\gamma_1 A \gamma_2 \rightarrow \gamma_1 \alpha \gamma_2
\]

where \( \gamma_1 \) and \( \gamma_2 \) range over \((N \cup \Sigma)^*\), \( A \) is in \( N \) and \( \alpha \) ranges over \((N \cup \Sigma)^+\).

Grammars in this form will be called type 1' grammars. A language is said to be of type 1' if there is a type 1' grammar which generates it. The class of type 1' languages will be referred to as \( L_{1'} \). We will use \( L_1 \) to refer to the class of context-sensitive languages as defined in the lecture notes.

(a) Prove that \( L_{1'} \subseteq L_1 \). \([5 \text{ marks}]\)

(b) Give constructions to show that \( L_{1'} \) is closed under set union and Kleene closure. \([16 \text{ marks}]\)

(c) Consider the following productions of a type 1 grammar \( G \):

\[
S \rightarrow aBC \mid aSBC \\
aB \rightarrow ab \\
bB \rightarrow bb
\]
Identify the production/s in $G$ which stop it from being a type 1′ grammar and modify the grammar so as to show that $L(G) \in \mathcal{L}_{1'}$.

[12 marks]

2. Consider the alphabet $\Sigma_1 = \{L, R\}$, where $L$ represents a unit step to the left and $R$ represents one to the right. A path is a (possibly empty) sequence of unit steps. Thus, for example, $LLR$ and $RRR$ are paths over the directions $L$ and $R$. A closed path is one which finishes exactly at the point of origin. Hence, for example, $LRRL$ and $\varepsilon$ are closed paths over $L$ and $R$.

For this question you may use the family of functions $\#_a$ (where $a$ is a terminal symbol) where $\#_a(s)$ returns the number of occurrences of $a$ in string $s$. Hence, for example, $\#_1(01110) = 3$.

(a) Using set comprehension, define the language $C_{\Sigma_1}$ which includes closed paths over the alphabet $\Sigma_1$.

[3 marks]

(b) Prove that $C_{\Sigma_1}$ is not in $\mathcal{L}_3$, the class of regular languages.

[14 marks]

(c) Design a context free grammar or a non-deterministic pushdown automaton to prove that $C_{\Sigma_1}$ is in $\mathcal{L}_2$.

[10 marks]

(d) Now consider the extended alphabet $\Sigma_2 = \{N, S, E, W\}$, representing unit steps in the north, south, east and west directions. Do you think $C_{\Sigma_2}$ is in $\mathcal{L}_2$, the class of context-free languages? Give reasons for your belief (not a proof).

[6 marks]

3. Recall that when drawing a non-deterministic pushdown automaton (NPDA), we use the following notation:

\[(a, b) / s\]

\[
\begin{array}{c}
A \\
\end{array} \\
\begin{array}{c}
\rightarrow \\
B
\end{array}
\]
The shown edge and labels mean that when the machine is in state $A$, with input $b \ (\in \Sigma \cup \{\epsilon\})$ and symbol $a \ (\in \Gamma)$ is on the top of the stack, it will go to state $B$, writing $s \ (\in \Gamma^*)$ onto the stack.

A two stack non-deterministic pushdown automaton (2-NPDA), uses a similar notation:

\[
\begin{array}{c}
A \quad (a_1, a_2), b / s_1, s_2 \\
\end{array} \quad B
\]

The edge now signifies that when the machine is in state $A$, with input $b \ (\in \Sigma \cup \{\epsilon\})$ and symbol $a_1 \ (\in \Gamma)$ is on the top of the first stack and $a_2 \ (\in \Gamma)$ is on the top of the second stack, it will go to state $B$, writing $s_1 \ (\in \Gamma^*)$ onto the first stack and $s_2 \ (\in \Gamma^*)$ onto the second.

In all other respects, 2-NPDA are the same as NPDA, with one initial state, a bottom of stack marker and a number of terminal states.

(a) Formalise the concept of a 2-NPDA. [5 marks]

(b) Define what one means by the *configuration* of a 2-NPDA. [5 marks]

(c) Construct a 2-NPDA which accepts language $L$:

\[
L \overset{\text{def}}{=} \{ a^i b^i c^i \mid i \in \mathbb{N} \}
\]

[18 marks]

(d) Hence *argue* (not prove) that 2-NPDA are more expressive than NPDA. [5 marks]

4. Prove that the following problem is NP-complete.

$\frac{1}{2}$-CLIQUE ($\frac{1}{2}$-CLI)

*Instance:* Given a graph $G$ with $n$ nodes.

*Question:* Does $G$ have a clique of size $\frac{n}{2}$?

Clearly state any results proved in the course notes which you use. [33 marks]
5. Consider the following general problem:

\textbf{\textit{i-SATISFIABILITY (i-SAT)}}

**INSTANCE:** Given a set of boolean variables $V$ and a set of clauses $C$, where each clause has exactly $i$ literals.

**QUESTION:** Is there a truth assignment of $V$ satisfying $C$?

(a) Prove that 1-SAT is in $P$. \hspace{1cm} [10 marks]

(b) Prove that 4-SAT is $NP$-complete. \hspace{1cm} [23 marks]

Clearly state any results proved in the course notes which you use.