Computer Graphics

(Geometrical Transformations)
Lecture 005

Vectors (Preliminary Material)

- A Vector is an n-tuple of real number where n=2 for 2D space and 3 for 3D space etc.
- Graphically we denote vectors as directed arrows. The length of the arrow defines the vector's magnitude.
- Vectors may be added to get: [x y z] + [x' y' z'] = [x+x',y+y',z+z']
- Vector may be multiplied by real numbers to get: s . [x y z] = [sx sy sz]

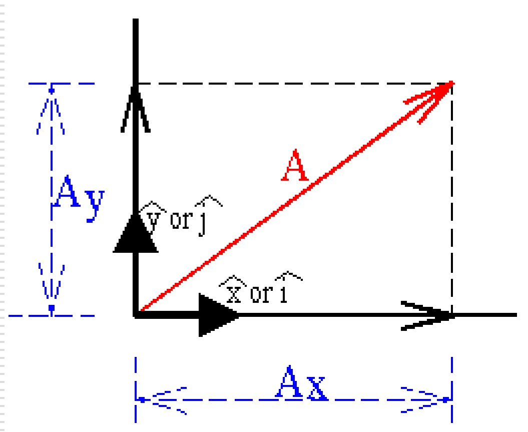
  Note that we are changing the magnitude and not the direction of the vector.
- Note also that many of the laws of ordinary algebra hold also for vector algebra.
  - Commutative Law for addition : A + B = B + A
  - Associative Law for addition : A + (B + C) = (A + B) + C
  - Commutative Law for multiplication : mA = Am
  - Associative Law for multiplication : (m+n)A = mA + nA
  - Distributive Law : m(A + B) = mA + mB

Vectors (Relationship with Coordinate System)

- Vectors can be related to the basic coordinate systems which we use by the introduction of what we call *unit vectors.*

- A unit vector is one which has a magnitude of 1 and is often indicated by putting a hat on top of the vector symbol.

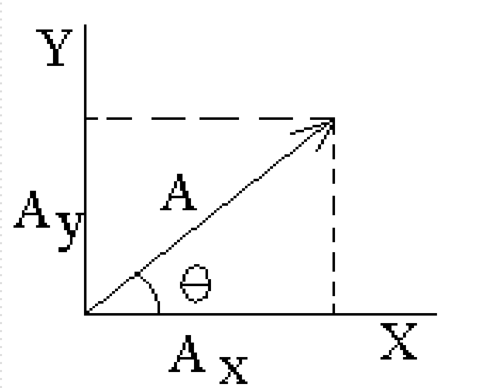
- Any two-dimensional vector can now be represented by employing multiples of the unit vectors.



Vectors (Relationship with Coordinate System)

- Often you will know the length (magnitude) of a vector and the direction of the vector. From these you will need to calculate the Cartesian components, that is, the x and y components.

- Ax = |A| cos theta
- Ay = |A| sin theta



- Note that 'obviously', the (x,y) Cartesian coordinates can be represented using polar coordinates.

Vectors (Two types of Vector Multiplication)

- Dot product is given by either:
  - |A| |B| cos theta , or
  - [x y z].[x’ y’ z’] = xx’ + yy’ + zz’

- The result of a dot product is a scalar.

- Uses of dot product:
  - *Magnitude* (length) of vector: |v| = v.v
  - *Angle* between 2 vectors: theta = cos - (v.w / |v|.|w|)
  - *Projection*: given a unit vector v and a vector w, the projection u of w in the direction of v is given by v.w

Vectors (Cross Product)

- If u and v are vectors in three dimensional space (only), then u x v is another three dimensional vector, where:
  - Length: |u x v| = |u||v| sin theta
  - Orientation: u x v is perpendicular to both u and v.

- The most important use of the cross product in computer graphics is to find a vector perpendicular to a plane.

- Note that there are two orientations perpendicular to u and v. The decision on which to choose depends on the right hand rule.

- Note that u x v and v x u point in opposite directions

2D Transformations (Translation, Scaling)

- The point (x,y) is translated to the point (x',y') by adding the translation distances (tx,ty).
- In matrix form we get [x' y'] = [x y] + [tx ty]
- The vertex (x,y) is scaled into the vertex (x', y') by multiplying it with the scaling factors sx and sy.
- Thus we get:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- If sx = sy the transformation is called a uniform scaling.
- If sx /= sy it is called a differential scaling

2D Transformations (Rotation)

- The point (x,y) or (r, Φ), is rotated anticlockwise about the origin by θ into the point (x’,y’) or (r, Φ+θ).

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Note that rotation occurs about the origin !!

- Have a look at the derivation on page 41.

Homogeneous Coordinates

- Translations, rotations and scaling can be combined together. However since translation is expressed as matrix addition, it is not, in general, possible to combine a set of operations into a single matrix operation.
- $V' = V + D$ (translation), $V' = S\,V$ (scaling), $V' = R\,V$ (rotation)
- Composition of transformations is always desirable is the same set of operations have to be applied to a list of position vectors.
- To enable the above transformations to be treated in the same way and combined, we use a system called *homogenous coordinates* which increases the dimensionality of the space. The practical reason for this is computer graphics is to enable us to include translation as matrix multiplication (rather than addition)

Homogeneous Coordinates (cont.)

- In a homogenous system a vertex V (x,y,z) is represented as V (w.X, w.Y, w.Z, w) for any scale factor w /= 0.

- The three-dimensional Cartesian coordinate representation is then:
  - X = X / w
  - Y = Y / w
  - Z = Z / w

- If we consider w to have the value 1 then the matrix representation of a point is: [x y z 1]

Transformations

- Consider the following pair of transformations:

Rotation followed by Translation

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & 0.5 & 0 & 0 \\ -0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 & 0 & 2 \\ -0.5 & 0.866 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation followed by Rotation

$$\begin{bmatrix} 0.866 & 0.5 & 0 & 0 \\ -0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 & 0 & 2.732 \\ -0.5 & 0.866 & 0 & 0.732 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- In general we get the following matrix where A is the net rotation and scaling while T gives the net translation

$$\begin{bmatrix} A11 & A12 & A13 & Tx \\ A21 & A22 & A23 & Ty \\ A31 & A32 & A33 & Tz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformations for Scaling, Translation and Rotation

- Refer to Page 50/51 on notes